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AN ASYMPTOTIC LOWER BOUND DYNAMIC BUCKLING ESTIMATE FOR IMPERFECT SYSTEMS UNDER IMPACT LOADING

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Abstract—A regular perturbation procedure is presented for the determination of lower bound dynamic buckling loads of discrete dissipative or nondissipative imperfect structural systems under impact loading. The procedure is based on the energy criterion for establishing the lower bound dynamic buckling estimate of autonomous systems without solving a set of differential equations of motion numerically presented by Kounadis and his associates. Attention is focused on the effect of imperfections on the lower bound estimate. As an expansion in terms of fractional powers of imperfection amplitude, the lower bound dynamic load is constructed by using an inverse expansion of the aforementioned procedure. Application of the procedure is demonstrated by two examples.
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1. INTRODUCTION

For a discrete nonlinear elastic structural system under impact loading, one can apply the law of impulse momentum to determine analytically initial conditions which are valid immediately after the impact. The response of the system after the impact is governed by a set of autonomous highly nonlinear ordinary differential equations and the exact value of dynamic buckling load λ_d of the system can be numerically obtained by integrating the equations of motion subject to the given initial conditions. However, the availability of high-speed computers and modern efficient computational techniques cannot overcome serious numerical difficulties in solving the nonlinear initial-value problem associated with the dynamic buckling response of the system. This is particularly true when an unbounded motion may be initiated after a very long period of time or in case of the existence of chaotic-like phenomena due to sensitivity to initial conditions or due to damping. In such cases the numerical computations that are used for the solution of the differential equations of motion may experience convergence difficulties. Therefore, it is highly desirable to supplement numerical simulation by other solutions such as approximate or lower/upper bound estimates for the dynamic buckling load. Such estimates not only help us to monitor the accuracy of the numerical algorithms but also provide fast and inexpensive approximations that can be very useful for structural design purposes (Kounadis, 1991a, b; Kounadis *et al.*, 1991; Kounadis, 1993a, b; Gantes and Kounadis, 1995; and Kounadis *et al.*, 1997).

Without solving the highly nonlinear system of ordinary differential equations, the lower-upper dynamic buckling estimates based on energy criteria for an initially imperfect discrete structural system under impact loading have been established by Kounadis (1993b) and Kounadis *et al.* (1997). Let λ be the loading parameter resulting after impact and ε a measure of the amplitude of a given imperfection pattern. We assume that the imperfect system under the same loading, λ, applied statically exhibits a limit point instability. Then the limit point load λ_s of this system is an upper bound of the dynamic buckling load λ_d. Determination of λ_s has been discussed by numerous investigators (see e.g. Thompson and Hunt, 1971; El Naschie, 1990; Wu and Wang, 1997). The lower bound dynamic buckling load $\tilde{\lambda}_d$ in addition to the exact one for the vanishing (but nonzero) damping and the corresponding displacement u_d are associated with an equilibrium point of the unstable post-buckling path where the total potential energy is equal to the initial kinetic energy.

Problems of determining lower dynamic buckling loads based on the energy criterion are essentially nonlinear, and an approximate analysis is required to solve them. There are at least two ways to obtain the solution: via approximate numerical methods (such as methods described by Simitses, 1990; Kounadis, 1993b; and Kounadis *et al.*, 1997), or via approximate analytical methods (such as a perturbation method to be discussed in the present paper). The latter has the advantage that it employs derivatives of the potential energy and the initial kinetic dynamic energy at the critical state in order to construct the solution of the system of nonlinear simultaneous equations resulting from the energy criterion, and these derivatives are also necessary to identify the lower bound dynamic buckling loads associated with unstable post-buckling equilibrium paths. We remark that it is impossible to find the lower bound estimate as a function of the imperfection amplitude ε, i.e. $\tilde{\lambda}_d = \tilde{\lambda}_d(\varepsilon)$ by perturbing the corresponding equations associated with the energy criterion in terms of integer powers of ε. This is due to the fact that in the analysis such a relation has an infinite slope at $\varepsilon = 0$ and the expansion cannot be made by the use of regular perturbation methods.

In this paper, one solution to this problem will be proposed. The perturbation is carried out with respect to another parameter ξ, the amplitude of projection of the corresponding displacement u_d on the buckling mode u_1 of the perfect system, and series are found in the form $\tilde{\lambda}_d = \tilde{\lambda}_d(\xi)$, $\varepsilon = \varepsilon(\xi)$ and $u_d = u_d(\xi)$. A unified presentation of the perturbation procedure in a form suitable for application to a wide variety of problems will be provided. Then the middle expansion is inverted to provide $\xi = \xi(\varepsilon)$, and the result is substituted into the two remaining series to produce the desired expansions. The three primary expansions are regular and can be obtained in a straight way. The final expansions are in terms of fractional powers of ε and have infinite slopes at $\varepsilon = 0$. Two examples are used to illustrate characteristics and effectiveness of the method.

2. BASIC FORMULAE

In this paper, the term "imperfect system" will be used repeatedly and will denote a more detailed structural model, which simulates some or all of the unintended deviations of the real system from a perfect model. These unintended deviations will be collectively denoted as imperfections. It is understood that the imperfections have been normalized so that, for zero magnitude of the imperfections, the imperfect system is reduced to the perfect one.

Consider an imperfect system under impact loading. Let the potential energy of the imperfect system after impact be given by $V(u, \lambda, w)$ where u denotes the additional displacement of the imperfect system from its initial configuration, λ the load parameter resulting after impact, and w the imperfection of the system. The load λ is considered as the main control parameter for the occurrence of static and dynamic bifurcation. Furthermore, let U and W denote, respectively, admissible function spaces of displacement and imperfection of the system. We introduce two inner products in the two spaces denoted by (u_1, u_2) for $u_i \in U (i = 1, 2)$ and $[w_1, w_2]$ for $w_i \in W (i = 1, 2)$ and the corresponding norms are $\|u\| = (u, u)^{1/2}$ for $u \in U$ and $|w| = [w, w]^{1/2}$ for $w \in W$, respectively. It is convenient to distinguish between the imperfection amplitude $\varepsilon = |w|$ and the normalized imperfection pattern $\bar{u} = w/|w|$. Thus the potential energy V may be written as $V(u, \lambda, \varepsilon\bar{u})$. Furthermore, for all λ and ε the potential energy V can always be chosen to obey the condition

$$V(0, \lambda, \varepsilon\bar{u}) = 0. \tag{1}$$

A crucial point for the dynamic analysis is the determination of the initial velocities. Their establishment can be achieved by employing (for the state immediately after the impact) the law of impulse momentum together with the vectorial kinematic relations between the velocities of the masses. One can further determine the initial kinetic energy as a function of the imperfection. We assume that, when no imperfection exists, the velocities of the system after the impact remain to be zero. Then by using the Taylor expansion, one can write the initial kinetic energy $K^0(\varepsilon\bar{u})$ of the imperfect system as

$$K^0(\varepsilon\bar{u}) = \frac{1}{2}K^0_{\varepsilon\varepsilon}(0)\varepsilon^2 + \frac{1}{6}K^0_{\varepsilon\varepsilon\varepsilon}(0)\varepsilon^3 + \cdots \quad (2)$$

where $K^0_{\varepsilon\varepsilon}(0)$ and $K^0_{\varepsilon\varepsilon\varepsilon}(0)$ depend on $\bar{u}$.

Throughout the paper, we will use the following notations: various functional (Fréchet) derivatives of the potential energy with respect to u and w are denoted by subscript $()_u$ and $()_w$, respectively, while the ordinary partial derivatives with respect to λ are denoted by a subscript $()_{\lambda}$, *et al.*

By eqn (1), the potential energy of the imperfect system is zero at the initial moment after impact. Consequently, according to the energy criterion for establishing lower bound dynamic buckling loads (Kounadis, 1993b; Kounadis *et al.*, 1997), the approximate dynamic buckling load can be found by requiring that the potential energy V at a certain unstable static equilibrium position be equal to the initial kinetic energy. That is that, for each magnitude ε of a given normalized imperfection pattern $\bar{u}$, the lower bound dynamic buckling load $\tilde{\lambda}_d$ can be found by solving the following nonlinear equations simultaneously (in unknowns u and λ):

$$V_u(u, \lambda, \varepsilon\bar{u}) = 0 \quad (3)$$

and

$$V(u, \lambda, \varepsilon\bar{u}) = K^0(\varepsilon\bar{u}) \quad (4)$$

subject to the condition that the static equilibrium position determined by the solution of eqns (3) and (4) is unstable.

We assume that, for the perfect system, there exists a trivial major equilibrium solution $u = u_0(\lambda) = 0$ as the load increases from zero, i.e.

$$V_u(0, \lambda, 0) = 0. \quad (5)$$

Let $\lambda = \lambda_c$ be the buckling load for the perfect system and it is assumed to be simple with corresponding buckling mode u_1 normalized by $\|u_1\| = 1$. In mathematical terms:

$$V^c_{uu}u_1 = 0 \quad (6)$$

where superscript c denotes the corresponding derivatives of potential energy function V calculated at $(u, \lambda, w) = (0, \lambda_c, 0)$. Then one has the decomposition of the space U:

$$U = N + N^{\perp}, \quad N = \{u \in U, u = \xi u_1, \xi \in R\}, \quad N^{\perp} = \{v \in U, (v, u_1) = 0\}. \quad (7)$$

We further assume that, when load parameter λ passes increasingly through its critical value λ_c, the trivial major equilibrium solution becomes unstable from stable (Thompson and Hunt, 1973; Budiansky, 1974):

$$V_{uu\lambda}u_1^2 < 0. \quad (8)$$

We also assume that the effect of imperfections is of the first order (Thompson and Hunt, 1973; Budiansky, 1974; Ikeda and Murota, 1990):

$$V^c_{uw}u_1\bar{u} \neq 0. \quad (9)$$

3. GENERAL DEVELOPMENT OF THE PERTURBATION PROCEDURE

In this section, first, the solutions to the system consisting of eqns (3) and (4) are established by using a regular perturbation procedure; second, the lower bound dynamic buckling loads are determined from the found solutions by discussion of their stabilities.

Traditionally, for each amplitude ε of a given normalized imperfection pattern $\bar{u}$, the approximate dynamic buckling strength of an imperfect system is treated by solving the system consisting of eqns (3) and (4) in unknown u and λ subject to the condition that the obtained static equilibrium position is unstable. However, as pointed out in the introduction, a perturbation expansion of the solution in terms of integer power of ε is impossible. Here, instead, it is suggested to solve the system consisting of eqns (3) and (4) with $u = \xi u_1 + v$, $v \in N^{\perp}$, and with v, λ, ε as unknowns for each ξ, the amplitude of the projection of the displacement u on the buckling mode u_1. Then the analysis of the approximate dynamic buckling strength in the present paper is directed exactly to determinations of dependences of u, λ, ε on ξ for the specified pattern of imperfection. The change of status of parameters will transform the original singular perturbation problem to a regular one as shown in the following.

We look for the solutions to the system consisting of eqns (3) and (4) in the following forms:

$$u = \xi u_1 + \xi^2 u_2 + \xi^3 u_3 + \xi^4 u_4 + \xi^5 u_5 + \cdots,$$

$$\lambda = \lambda_c + \lambda_1 \xi + \lambda_2 \xi^2 + \lambda_3 \xi^3 + \lambda_4 \xi^4 + \cdots,$$

$$\varepsilon = \varepsilon_2 \xi^2 + \varepsilon_3 \xi^3 + \varepsilon_4 \xi^4 + \varepsilon_5 \xi^5 + \cdots \qquad (10\text{a,b,c})$$

where $(u_1, u_i) = 0 (i = 2, 3, \ldots)$.

Substituting eqns (10a–c) into eqns (3) and (4), expanding them, respectively, into series of ξ, we obtain

$$\xi^2 (V^c_{uu} u_2 + \varepsilon_2 V^c_{uw} \bar{u} + \lambda_1 V^c_{uu\lambda} u_1 + \tfrac{1}{2} V^c_{uuu} u_1^2)$$
$$+ \xi^3 [V^c_{uu} u_3 + \varepsilon_3 V^c_{uw} \bar{u} + \lambda_2 V^c_{uu\lambda} u_1 + \varepsilon_2 (\lambda_1 V^c_{uw\lambda} \bar{u} + V^c_{uuw} u_1 \bar{u})$$
$$+ \lambda_1 V^c_{uu\lambda} u_2 + \tfrac{1}{2} \lambda_1^2 V^c_{uu\lambda\lambda} u_1 + V^c_{uuu} u_1 u_2 + \tfrac{1}{2} \lambda_1 V^c_{uuu\lambda} u_1^2 + \tfrac{1}{6} V^c_{uuuu} u_1^3] + \text{h.o.t.} = 0 \qquad (11)$$

and

$$\xi^3 (\varepsilon_2 V^c_{uw} u_1 \bar{u} + \tfrac{1}{2} \lambda_1 V^c_{uu\lambda} u_1^2 + \tfrac{1}{6} V^c_{uuu} u_1^3)$$
$$+ \xi^4 [\varepsilon_3 V^c_{uw} u_1 \bar{u} + \tfrac{1}{2} \lambda_2 V^c_{uu\lambda} u_1^2 + \varepsilon_2 (V^c_{uw} u_2 \bar{u} + \lambda_1 V^c_{uw\lambda} u_1 \bar{u} + \tfrac{1}{2} V^c_{uuw} u_1^2 \bar{u})$$
$$+ \lambda_1 V^c_{uu\lambda} u_1 u_2 + \tfrac{1}{2} V^c_{uu} u_2^2 + \tfrac{1}{4} \lambda_1^2 V^c_{uu\lambda\lambda} u_1^2 + \tfrac{1}{2} V^c_{uuu} u_1^2 u_2$$
$$+ \tfrac{1}{6} \lambda_1 V^c_{uuu\lambda} u_1^3 + \tfrac{1}{24} V^c_{uuuu} u_1^4 - \tfrac{1}{2} K^0_{\varepsilon\varepsilon}(0) \varepsilon_2^2] + \text{h.o.t.} = 0 \qquad (12)$$

Note that the coefficient of $\xi^i (i \geqslant 2)$ in eqn (11) has been derived by inserting eqns (10a–c) into eqn (3), differentiating i times with respect to ξ, making use of eqns (5) and (6), setting $\xi = 0$ and then dividing by $i!$. The coefficient of $\xi^j (j \geqslant 3)$ in eqn (12) has been similarly obtained.

The coefficients of $\xi^2, \xi^3, \ldots$ in eqn (11) as well as those of $\xi^3, \xi^4, \ldots$ in eqn (12) must vanish separately. We divide the resulted equations into the following groups:

$$V^c_{uu} u_2 + \varepsilon_2 V^c_{uw} \bar{u} + \lambda_1 V^c_{uu\lambda} u_1 + \tfrac{1}{2} V^c_{uuu} u_1^2 = 0,$$

$$(u_2, u_1) = 0,$$

$$\varepsilon_2 V^c_{uw} u_1 \bar{u} + \tfrac{1}{2} \lambda_1 V^c_{uu\lambda} u_1^2 + \tfrac{1}{6} V^c_{uuu} u_1^3 = 0 \qquad (13\text{a,b,c})$$

and

$$V^c_{uu}u_3+\varepsilon_3 V^c_{uw}\bar{u}+\lambda_2 V^c_{uu\lambda}u_1+\varepsilon_2(\lambda_1 V^c_{uw\lambda}\bar{u}+V^c_{uuw}u_1\bar{u})$$
$$+\lambda_1 V^c_{uu\lambda}u_2+\tfrac{1}{2}\lambda_1^2 V^c_{uu\lambda\lambda}u_1+V^c_{uuu}u_1u_2+\tfrac{1}{2}\lambda_1 V^c_{uuu\lambda}u_1^2+\tfrac{1}{6}V^c_{uuuu}u_1^3=0,$$
$$(u_3,u_1)=0,$$
$$\varepsilon_3 V^c_{uw}u_1\bar{u}+\tfrac{1}{2}\lambda_2 V^c_{uu\lambda}u_1^2+\varepsilon_2(V^c_{uw}u_2\bar{u}+\lambda_1 V^c_{uw\lambda}u_1\bar{u}+\tfrac{1}{2}V^c_{uuw}u_1^2\bar{u})$$
$$+\lambda_1 V^c_{uu\lambda}u_1u_2+\tfrac{1}{2}V^c_{uu}u_2^2+\tfrac{1}{4}\lambda_1^2 V^c_{uu\lambda\lambda}u_1^2+\tfrac{1}{2}V^c_{uuu}u_1^2u_2$$
$$+\tfrac{1}{6}\lambda_1 V^c_{uuu\lambda}u_1^3+\tfrac{1}{24}V^c_{uuuu}u_1^4-\tfrac{1}{2}K^0_{\varepsilon\varepsilon}(0)\varepsilon_2^2=0 \quad \text{(14a,b,c)}$$

and so on.

Taking an inner product in the two sides of eqn (13a) with u_1, and using eqn (6) we obtain:

$$\varepsilon_2 V^c_{uw}u_1\bar{u}+\lambda_1 V^c_{uu\lambda}u_1^2+\tfrac{1}{2}V^c_{uuu}u_1^3=0. \quad (15)$$

Making use of inequalities (8) and (9), we can solve the linear system consisting of eqns (13c) and (15) in unknowns ε_2 and λ_1, and obtain

$$\lambda_1=-\frac{2V^c_{uuu}u_1^3}{3V^c_{uu\lambda}u_1^2} \quad (16)$$

and

$$\varepsilon_2=\frac{V^c_{uuu}u_1^3}{6V^c_{uw}u_1\bar{u}}. \quad (17)$$

The substitution of eqns (16) and (17) into eqns (13a, b) leads to equation governing u_2 and its constraint condition, and the solution to this kind of system can be achieved by using a method described in Wu (1997b). Inserting the found u_2, eqns (16) and (17) into eqns (14a–c), respectively, we get a linear system in unknowns u_3, ε_3 and λ_2 and its solution can be obtained in the same method like that of eqns (13a–c). We here list the results of λ_2 and ε_3 as follows (note that $V^c_{uu}u_2^2+\varepsilon_2 V^c_{uw}u_2\bar{u}+\lambda_1 V^c_{uu\lambda}u_1u_2+\frac{1}{2}V^c_{uuu}u_1^2u_2=0$ which can be derived by taking inner product in the two sides of eqn (13a) with u_2, thus the term $V^c_{uu}u_2^2$ can be canceled):

$$\lambda_2=\frac{1}{V^c_{uu\lambda}u_1^2}\Big[\varepsilon_2(V^c_{uw}u_2\bar{u}-V^c_{uuw}u_1^2\bar{u})-\lambda_1 V^c_{uu\lambda}u_1u_2-\frac{3}{2}V^c_{uuu}u_1^2u_2$$
$$-\frac{1}{2}\lambda_1^2 V^c_{uu\lambda\lambda}u_1^2-\frac{2}{3}\lambda_1 V^c_{uuu\lambda}u_1^3-\frac{1}{4}V^c_{uuuu}u_1^4-K^0_{\varepsilon\varepsilon}(0)\varepsilon_2^2\Big] \quad (18)$$

and

$$\varepsilon_3=\frac{1}{V^c_{uw}u_1\bar{u}}\Big[-\varepsilon_2(V^c_{uw}u_2\bar{u}+\lambda_1 V^c_{uw\lambda}u_1\bar{u})+\frac{1}{2}V^c_{uuu}u_1^2u_2$$
$$+\frac{1}{6}\lambda_1 V^c_{uuu\lambda}u_1^3+\frac{1}{12}V^c_{uuuu}u_1^4+K^0_{\varepsilon\varepsilon}(0)\varepsilon_2^2\Big]. \quad (19)$$

Furthermore, u_3 can be solved from eqns (14a, b) with ε_2, λ_1, ε_3 and λ_2 given in eqns (16)–(19).

The expressions for the successive coefficients $u_i(i > 3)$, $\lambda_j(j > 2)$ and $\varepsilon_k(k > 3)$ in eqns (10a–c) can be found in a straightforward way as u_2, λ_1, and ε_2. These expressions are lengthy and tedious, and thus will be omitted.

If $V^c_{uuu}u_1^3 = 0$, we derive from eqns (16) and (17) $\lambda_1 = 0$, $\varepsilon_2 = 0$. Furthermore, utilizing the expressions (2) and (10c) and the perturbations process, we know that, in the perturbation expansions (10a–c), only the coefficients λ_i, u_{i+1}, and $\varepsilon_{i+1}(i \geqslant 4)$ have relations to the initial kinetic energy. We will illustrate this point by an example in the next section.

Finally, by utilizing the same discussion on the stability of equilibrium positions as that in Wu (1997a) and using a method of reversing an incomplete series developed by Hunt (1971), we can express the lower bound dynamic buckling load in terms of the amplitude of a given normalized imperfection pattern. For the asymmetric case of bifurcation ($V^c_{uuu}u_1^3 \neq 0$), we have

$$\tilde{\lambda}_d = \lambda_c - \operatorname{sign}(\lambda_1)\lambda_1\left(\frac{1}{\varepsilon_2}\right)^{1/2}\varepsilon^{1/2} + \left(\frac{\lambda_2}{\varepsilon_2} - \frac{\lambda_1\varepsilon_3}{2\varepsilon_2^2}\right)\varepsilon + O(|\varepsilon|^{3/2}) \tag{20}$$

where $\operatorname{sign}(\varepsilon_2) = 1$ and only the terms with order equal or higher than ε have concerns with the initial kinetic energy. For the symmetric case of bifurcation ($V^c_{uuu}u_1^3 = 0$), the expression becomes

$$\tilde{\lambda}_d = \lambda_c + \lambda_2\left(\frac{1}{\varepsilon_3}\right)^{2/3}\varepsilon^{2/3} + \left(\frac{\lambda_3}{\varepsilon_3} - \frac{2\lambda_2\varepsilon_4}{3\varepsilon_3^2}\right)\varepsilon + \left(\frac{1}{\varepsilon_3}\right)^{1/3}\left(\frac{\lambda_4}{\varepsilon_3} - \frac{\lambda_3\varepsilon_4}{\varepsilon_3^2} - \frac{2\lambda_2\varepsilon_5}{3\varepsilon_3^2} + \frac{7\lambda_2\varepsilon_4^2}{9\varepsilon_3^3}\right)\varepsilon^{4/3} + O(|\varepsilon|^{5/3}) \tag{21}$$

where $\lambda_2 < 0$ and only the terms with order equal or higher than $\varepsilon^{4/3}$ have relations to the initial kinetic energy.

4. EXAMPLE

In this section, we will present two examples to illustrate application of the method.

4.1. *Example* 1 : *a single-degree-of-freedom cantilevered model under impact loading*

The first example is the initially imperfect single-degree-of-freedom model under impact loading illustrated in Fig. 1. The model is an extension of Kounadis and Raftoyiannis

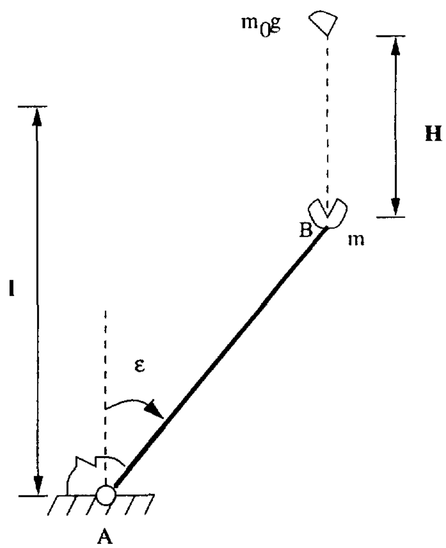

Fig. 1. Single-degree-of-freedom imperfect model under impact loading.

(1990) and Kounadis (1993b) by replacing the original quadratic nonlinear rotational spring with the cubic nonlinear one that exhibits a softening behavior described by the equation $M_r = k(\theta - \theta^3)$. A body of weight $m_0 g$ (g is the gravitational acceleration) falling from a height H strikes centrally the tip mass m with initial velocity $v_0 = \sqrt{2gH}$. We assume a completely plastic impact occurs; thus both the striking body and the tip mass attain immediately after the impact the same velocity and do not separate thereafter. It is assumed that the response of the model after impact remains elastic throughout deformation.

The equilibrium equation of the system after the impact is

$$\theta - \theta^3 - \lambda \sin(\theta + \varepsilon) = 0. \tag{22}$$

where ε and θ are the initial and incremental angle of deformation, respectively, $\lambda = m_0 g l / k$, k is the linear spring component; and condition (4) becomes

$$\tfrac{1}{2}\theta^2 - \tfrac{1}{4}\theta^4 - \lambda[\cos\varepsilon - \cos(\theta + \varepsilon)] = \tfrac{1}{2}\mu_0^2 \bar{v}_0^2 \sin^2 \varepsilon. \tag{23}$$

where $M = m_0 + m$, $\mu_0 = m_0/M$ and $\bar{v}_0 = v_0\sqrt{M/k}$.

It is easy to find that the buckling load of the perfect system is $\lambda_c = 1$. Due to the expressions of eqns (22) and (23), we look for their solutions in the following form:

$$\begin{aligned} \lambda &= 1 + \lambda_2\theta^2 + \lambda_4\theta^4 + \cdots, \\ \varepsilon &= \varepsilon_3\theta^3 + \varepsilon_5\theta^5 + \cdots. \end{aligned} \tag{24a,b}$$

Substituting eqns (24a,b) into eqns (22) and (23), and repeating the process as in deriving the perturbation equations in eqns (13a–c) and (14a–c), we get the following systems of equations:

$$\begin{aligned} &\varepsilon_3 + \lambda_2 + \tfrac{5}{6} = 0, \\ &\varepsilon_3 + \tfrac{1}{2}\lambda_2 + \tfrac{5}{24} = 0 \end{aligned} \tag{25}$$

and

$$\begin{aligned} &\varepsilon_5 + \lambda_4 - \tfrac{1}{2}\varepsilon_3 + \tfrac{1}{120} + \lambda_2(\varepsilon_3 - \tfrac{1}{6}) = 0, \\ &\varepsilon_5 + \tfrac{1}{2}\lambda_4 - \tfrac{1}{6}\varepsilon_3 + \tfrac{1}{720} + \lambda_2(\varepsilon_3 - \tfrac{1}{24}) + \tfrac{1}{2}\mu_0^2\bar{v}_0^2\varepsilon_3^2 = 0. \end{aligned} \tag{26}$$

Solving two systems (25) and (26), we obtain

$$\lambda_2 = -\tfrac{5}{4}, \quad \varepsilon_3 = \tfrac{5}{12} \tag{27}$$

and

$$\lambda_4 = -\tfrac{7}{144} + \tfrac{25}{144}\mu_0^2\bar{v}_0^2, \quad \varepsilon_5 = \tfrac{101}{180} - \tfrac{25}{144}\mu_0^2\bar{v}_0^2. \tag{28}$$

Finally, due to $\lambda_2 < 0$, from eqns (21), (24a, b), (27) and (28), one arrives at

$$\tilde{\lambda}_d = 1 - \tfrac{5}{4}(\tfrac{12}{5}\varepsilon)^{2/3} + (\tfrac{12}{5})^{1/3}(\tfrac{773}{300} - \tfrac{5}{12}\mu_0^2\bar{v}_0^2)\varepsilon^{4/3} + O(\varepsilon^2). \tag{29}$$

We remark that eqn (29) gives an asymptotic expression of the exact dynamic buckling load for the undamped model under the impact loading.

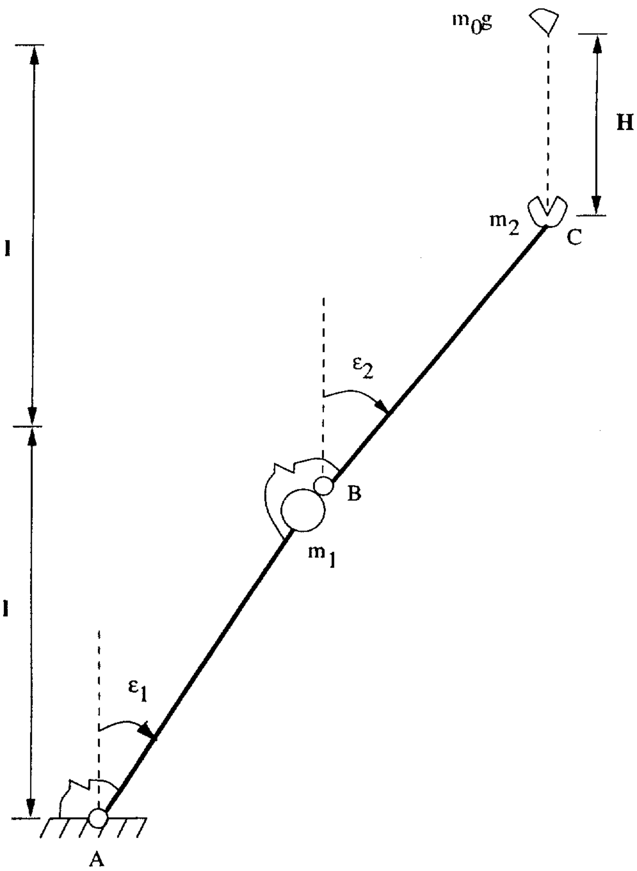

Fig. 2. Ziegler's geometrically imperfect model under impact loading.

4.2. *Example* 2: *Ziegler's two-degree-of-freedom model under impact loading*

The second example is the Ziegler's two-degree-of-freedom cantilevered model under impact loading investigated by Kounadis (1993b) and shown in Fig. 2. A body of weight m_0g falling from a height H strikes centrally the tip mass m_2 with initial velocity $v_0 = \sqrt{2gH}$. A completely plastic impact is also assumed.

For the state immediately after the impact (with masses m_1 and $M = m_0+m_2$) which corresponds to the initial position specified by the displacements (geometric imperfections) ε_1 and ε_2, one can apply the law of the impulse momentum to derive the initial velocities. We adopt a method which is different from Kounadis's (1993b) to obtain the initial angular velocities of the links AB and BC.

Applying the law of impulse momentum to the total system in the normal of AB and to the subsystem consisting of link BC and the new tip mass M in the normal of BC, respectively, one obtains relations between the initial nondimensional angular velocities $\dot{\Theta}_1(0)$ and $\dot{\Theta}_2(0)$ $(\mu_0 := m_0/M, \mu_1 := m_1/M, \bar{v}_0 := v_0\sqrt{M/k})$:

$$(1+\mu_1)\dot{\Theta}_1(0)+\cos(\varepsilon_2-\varepsilon_1)\dot{\Theta}_2(0) = -\mu_0\bar{v}_0\sin\varepsilon_1,$$
$$\cos(\varepsilon_2-\varepsilon_1)\dot{\Theta}_1(0)+\dot{\Theta}_2(0) = -\mu_0\bar{v}_0\sin\varepsilon_2. \quad (30)$$

Solution of the above linear system of equations is given by

$$\dot{\Theta}_1(0) = -\frac{\mu_0\bar{v}_0\cos\varepsilon_2\sin(\varepsilon_1-\varepsilon_2)}{\mu_1+\sin^2(\varepsilon_1-\varepsilon_2)},$$
$$\dot{\Theta}_2(0) = -\frac{\mu_0\bar{v}_0[\cos\varepsilon_1\sin(\varepsilon_2-\varepsilon_1)+\mu_1\sin\varepsilon_2]}{\mu_1+\sin^2(\varepsilon_1-\varepsilon_2)}. \quad (31)$$

Based on eqn (31), the nondimensional initial kinetic energy is

$$K^0(\varepsilon\bar{u}) = \frac{1}{2}[(1+\mu_1)\dot{\Theta}_1^2(0)+\dot{\Theta}_2^2(0)+2\dot{\Theta}_1(0)\dot{\Theta}_2(0)\cos(\varepsilon_2-\varepsilon_1)]$$

$$= \frac{\mu_0^2 \bar{v}_0^2}{2\mu_1}[(\theta_{10}-\theta_{20})^2+\mu_1\theta_{20}^2]\varepsilon^2+O(\varepsilon^3) \tag{32a}$$

where the imperfection

$$w = (\varepsilon_1, \varepsilon_2) = \varepsilon\bar{u}, \quad \bar{u} = (\theta_{10}, \theta_{20}), \quad \theta_{10}^2+\theta_{20}^2 = 1. \tag{32b}$$

The exact expression for the nondimensional total potential energy after the impact is given by

$$V(\theta_1,\theta_2,\lambda,\varepsilon_1,\varepsilon_2) = \tfrac{1}{2}\theta_1^2+\tfrac{1}{3}\delta_1\theta_1^3+\tfrac{1}{2}(\theta_2-\theta_1)^2+\tfrac{1}{3}\delta_2(\theta_2-\theta_1)^3$$
$$-\lambda[\cos\varepsilon_1-\cos(\theta_1+\varepsilon_1)+\cos\varepsilon_2-\cos(\theta_2+\varepsilon_2)] \tag{33}$$

where θ_1 and θ_2 are the incremental angle of deformation of the system, k the linear spring component common for both springs and $\delta_i(i = 1,2)$ are the nonlinear components of the corresponding quadratic springs, $\delta_i > 0(<0)$ express the corresponding spring is of hard(soft) type, $\lambda = m_0 gl/k$. Thus the equilibrium equation of the system is as follows:

$$\theta_1+\delta_1\theta_1^2-(\theta_2-\theta_1)-\delta_2(\theta_2-\theta_1)^2-\lambda\sin(\theta_1+\varepsilon_1) = 0,$$
$$\theta_2-\theta_1+\delta_2(\theta_2-\theta_1)^2-\lambda\sin(\theta_2+\varepsilon_2) = 0. \tag{34}$$

The buckling load as well as the buckling mode of the corresponding perfect system are

$$\lambda_c = \frac{3-\sqrt{5}}{2}, \quad u_1 = \begin{pmatrix}\theta_{11}\\ \theta_{12}\end{pmatrix} = \begin{pmatrix}\dfrac{\sqrt{5}-1}{2}\\ 1\end{pmatrix}. \tag{35}$$

From eqns (32b), (33) and (35),

$$V_{uu\lambda}^c u_1^2 = -\frac{5-\sqrt{5}}{2},$$

$$V_{uw}^c u_1\bar{u} = -\frac{3-\sqrt{5}}{4}[(\sqrt{5}-1)\theta_{10}+2\theta_{20}],$$

$$V_{uuu}^c u_1^3 = (2\sqrt{5}-4)\delta_1+(18-8\sqrt{5})\delta_2. \tag{36}$$

Substitutions of eqn (36) into eqns (16) and (17) lead to

$$\lambda_1 = \frac{1}{15}[(6\sqrt{5}-10)\delta_1+(50-22\sqrt{5})\delta_2],$$

$$\varepsilon_2 = -\frac{(\sqrt{5}-1)\delta_1+(7-3\sqrt{5})\delta_2}{3[(\sqrt{5}-1)\theta_{10}+2\theta_{20}]}. \tag{37}$$

In order to obtain λ_2 and ε_3, we must solve $u_2 = (\theta_{21}, \theta_{22})'$. From eqns (13a, b) and (34), u_2 satisfies the following system of equations:

$$\begin{pmatrix} 2-\lambda_c & -1 \\ -1 & 1-\lambda_c \end{pmatrix}\begin{pmatrix} \theta_{21} \\ \theta_{22} \end{pmatrix} - \lambda_c \varepsilon_2 \begin{pmatrix} \theta_{10} \\ \theta_{20} \end{pmatrix} - \lambda_1 \begin{pmatrix} \theta_{11} \\ \theta_{12} \end{pmatrix} + \begin{pmatrix} \delta_1 \theta_{11}^2 - \delta_2(\theta_{12}-\theta_{11})^2 \\ \delta_2(\theta_{12}-\theta_{11})^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{38a}$$

and constraint condition

$$\frac{\sqrt{5}-1}{2}\theta_{21} + \theta_{22} = 0. \tag{38b}$$

The solution of system (38a, b) is given by

$$\theta_{21} = \frac{[(4\sqrt{5}-10)\delta_1 + (40-18\sqrt{5})\delta_2]\theta_{10}}{15[(\sqrt{5}-1)\theta_{10}+2\theta_{20}]} - \frac{(\sqrt{5}+1)\delta_1 + (11\sqrt{5}-27)\delta_2}{30} \tag{39a}$$

and

$$\theta_{22} = -\frac{\sqrt{5}-1}{2}\theta_{21}. \tag{39b}$$

From eqns (32a) and (33),

$$\begin{aligned}
K^0_{\varepsilon\varepsilon}(0) &= \frac{\mu_0^2 \bar{v}_0^2}{\mu_1}[(\theta_{10}-\theta_{20})^2 + \mu_1\theta_{20}^2], \\
V^c_{uw} u_2 \bar{u} &= -\lambda_c(\theta_{21}\theta_{10} + \theta_{22}\theta_{20}), \\
V^c_{uw\lambda} u_1 \bar{u} &= -(\theta_{11}\theta_{10} + \theta_{12}\theta_{20}), \\
V^c_{uuu} u_1^2 u_2 &= 2\delta_1\theta_{11}^2\theta_{21} + 2\delta_2(\theta_{12}-\theta_{11})^2(\theta_{22}-\theta_{21}) \\
V^c_{uuu\lambda} u_1^3 &= 0, \\
V^c_{uuuu} u_1^4 &= \lambda_c(\theta_{11}^4 + \theta_{12}^4), \\
V^c_{uu\lambda\lambda} u_1^2 &= 0, \\
V^c_{uu\lambda} u_1 u_2 &= -\theta_{11}\theta_{21} - \theta_{12}\theta_{22}, \\
V^c_{uuw} u_1^2 \bar{u} &= 0.
\end{aligned} \tag{40}$$

Substitutions of eqns (35)–(37), (39a, b) and (40) into eqns (18) and (19) lead to

$$\lambda_2 = \left[\left(\frac{3\sqrt{5}-5}{30}\delta_1 + \frac{25-11\sqrt{5}}{30}\delta_2\right)\frac{(\sqrt{5}-1)\theta_{20}-2\theta_{10}}{(\sqrt{5}-1)\theta_{10}+2\theta_{20}} + \frac{3(5-\sqrt{5})}{10}\delta_1 - \frac{3(5-3\sqrt{5})}{10}\delta_2\right]\theta_{21} + \frac{15-6\sqrt{5}}{20} + \frac{5+\sqrt{5}}{10}K^0_{\varepsilon\varepsilon}(0)\varepsilon_2^2 \tag{41}$$

and

$$\varepsilon_3 = -\frac{1}{(\sqrt{5}-1)\theta_{10}+2\theta_{20}}\left\{-\frac{[(\sqrt{5}-1)\delta_1 + (7-3\sqrt{5})\delta_2][4\sqrt{5}\delta_1 + (20-8\sqrt{5})\delta_2]}{45} + [(2\theta_{10} - \sqrt{5}\theta_{20} + \theta_{20})\varepsilon_2 + 2\delta_1 - (\sqrt{5}-1)\delta_2]\theta_{21} + \frac{3-\sqrt{5}}{4} + (3+\sqrt{5})K^0_{\varepsilon\varepsilon}(0)\varepsilon_2^2\right\}. \tag{42}$$

Table 1. Lower-upper bound dynamic buckling estimates $\tilde{\lambda}_d$ and λ_s for different values of the imperfection amplitude, ε, for the case of impact loading with $\delta_1 = -2.5$, $\delta_2 = -0.75$, $\mu_0 = 0.1$, $\mu_1 = 1.8$, $\bar{v}_0 = 4.0$ and $\bar{u} = (\theta_{10}, \theta_{20})$

	$\tilde{\lambda}_d$	λ_s	$\tilde{\lambda}_d$	λ_s	$\tilde{\lambda}_d$	λ_s
ε	(1.0, 0.0)		(−0.707, 0.707)		(0.526, 0.851)	
0.001	0.361432	0.364289	0.368356	0.370281	0.354152	0.357582
0.005	0.335712	0.342440	0.351208	0.355838	0.320703	0.327441
0.01	0.316193	0.326068	0.338125	0.345016	0.296313	0.304856
0.02	0.288230	0.302914	0.319279	0.329711	0.262806	0.272916
0.03	0.266488	0.285147	0.304545	0.317967	0.237878	0.248407
0.04	0.247965	0.270170	0.291940	0.308066	0.217392	0.227746
0.05	0.231501	0.256974	0.280695	0.299343	0.199744	0.209543
0.08	0.189585	0.223862	0.251904	0.277456	0.157111	0.163866
0.1	0.165581	0.205200	0.235316	0.265120	0.134124	0.138122

Finally, one can obtain the asymptotic expansion (20) in terms of ε of the lower bound dynamic buckling load where λ_1, ε_2, λ_2 and ε_3 are given in eqns (37), (41) and (42).

For the present example, the asymptotic lower bound dynamic buckling loads and static buckling loads [obtained according to Wu and Wang (1997)] for three different normalized imperfection patterns $\bar{u} = (1.0, 0.0)$, $(-0.707, 0.707)$ and $(0.526, 0.851)$ are listed in Table 1. As expected, the lower-upper dynamic buckling estimates decrease with the increase of imperfection amplitude ε, and along with $\bar{u} = (0.526, 0.851)$ which has the shape of the buckling mode u_1 the lower-upper estimates decrease most rapidly.

5. CONCLUSIONS

Without solving the system of a differential equation of motion numerically, the present paper provides a regular perturbation approach to analysis of lower bound dynamic buckling loads for discrete imperfect structural systems under impact loading. The approach is based on the energy criteria for establishing the lower bound dynamic buckling estimate of autonomous systems presented by Kounadis and his associates. The relationship between the lower bound dynamic buckling load and a measure of the amplitude of the imperfection has been obtained by using an inverse expansion of the proposed regular perturbation procedure. With this relationship, the effects of various imperfection parameters on the lower bound dynamic buckling loads for autonomous systems under impact loading can be evaluated.

Here we applied the procedure to two examples. The results illustrate the characteristics and the effectiveness of the approach.

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